Code AS-634

Pre Ph.D. (Course Work) Examination 2014

Mathematics

(Approximation Theory) Alternate answers may follow.

1 (i) Clearly $d(x,x)=0$, $d(x,y)=1>0$ if $x \neq y$

$d(x,y)=d(y,x)=1$, $d(x,y)=1 \leq d(x,z)+d(z,y)$

Hence (X,d) is a metric space

(ii) Let K be a compact set in a metric space X $(p \in X)$, there corresponds a point in K of minimum distance from p

(iii) A mapping ϕ from one metric space to another metric space is said to be continuous at a point x if for every sequence $x_n \longrightarrow x$, it follows that $\phi(x_n) \longrightarrow \phi(x)$.

(iv) A mapping $\phi : [0,1] \longrightarrow [a,b]$ defined by $\phi(x) = bx+(1-x)a$. Then ϕ is continuous.

(v) If a sequence of vectors $\{f_1, f_2, \ldots\}$ has the Cauchy Property, $\lim\limits_{m,n\to\infty} \|f_n - f_m\| = 0$, then there exists a vector g such that $\lim\limits_{n\to\infty} \|f_n - g\| = 0$

(vi) Since $\|f\| \leq \|f-g\| + \|g\|$ so that $\|f\| - \|g\| \leq \|f\| + \|g\|$. By interchanging f and g, we obtain $\big| \|f\| - \|g\| \big| \leq \|f\| - \|g\|$ which proves the continuity of f.

(vii) $C[a,b]$ is a space of real valued continuous functions which is complete, so it is a Banach space and norm is defined as $\|f\| = \max\limits_{a \leq x \leq b} |f(x)|$

(viii) Yes, every finite-dimensional space is complete.

(ix) In an space of n tuples if $f=[\xi_1, \xi_2, \ldots, \xi_n]$ and $g=[\eta_1, \eta_2, \ldots \eta_n]$, it is defined as

$\langle f, g \rangle = \xi_1\eta_1 + \xi_2\eta_2 + \cdots + \xi_n\eta_n$

Clearly $\langle f, f \rangle > 0$ unless $f=0$

$\langle f, g \rangle = \langle g, f \rangle$

~~P.T.O.~~

$\langle f, \lambda g + \mu h \rangle = \lambda \langle f, g \rangle + \mu \langle f, h \rangle$

So it is an inner product space.

(x) Symmetry $\langle f, g \rangle = \langle g, f \rangle$

Linearity $\langle f, \lambda g + \mu h \rangle = \lambda \langle f, g \rangle + \mu \langle f, h \rangle$

where λ and μ are scalars. and $f, g, h \in E$ (a linear space)

(xi) The orthonormal property of set of vectors $\{f_1, f_2, \cdots\}$ is expressed by

$$\langle f_n, f_m \rangle = \delta_{nm} \quad (n, m = 1, 2, 3, \cdots)$$

(xii) Clearly $\langle f_0, f_1 \rangle = \int_0^\pi 1 . \cos x \, dx = 0$

$$\langle f_1, f_2 \rangle = \int_0^\pi \cos x \cos 2x \, dx = 0$$

$$\langle f_2, f_0 \rangle = \int_0^\pi \cos 2x \, dx = 0$$

Hence set of functions $\{f_0, f_1, f_2\}$ are orthogonal with respect to given $\langle f, g \rangle = \int_0^\pi f(x) g(x) \, dx$

(xiii) A set K is convex if whenever f_i ~~and g belongs to~~ $\in K$ $(i = 1, 2, \cdots n)$, $\theta_i \geq 0 \Rightarrow \sum_{i=1}^{m} \theta_i f_i \in K$

(xiv) A function ϕ defined on convex set is convex if $\phi(\lambda x + \mu y) \leq \lambda \phi(x) + \mu \phi(y)$

(xv) A biquadratic polynomial $y = ax^4 + bx^3 + cx^2 + dx + e$ can be made to pass through any five points having distinct abscissas.

(xvi) Let $p(x) = c_0 + c_1 x + c_2 x^2$ passing through points $(1, 2)$, $(2, -1)$ and $(3, 1)$ satisfy

$$c_0 + c_1 + c_2 = 2$$
$$c_0 + 2c_1 + 4c_2 = -1$$
$$c_0 + 3c_1 + 9c_2 = 1$$

$$\Longrightarrow p(x) = 10 - \frac{21}{2} x + \frac{5}{2} x^2$$

(2)' Let K be a such a set. Take $x_1, x_2, \cdots \in K$ such that $\lim_{i\to\infty} \|x_i\| = d = \inf_{x\in K} \|x\|$. Then by the Parallelogram law $\|x_i - x_j\|^2 = 2\|x_i\|^2 + 2\|x_j\|^2 - 4\|\frac{1}{2}(x_i+x_j)\|^2$.
Since K is convex, $\frac{1}{2}(x_i+x_j) \in K$ so that $\|\frac{1}{2}(x_i+x_j)\| \geq d$.
Hence $\|x_i - x_j\|^2 \leq 2\|x_i\|^2 + 2\|x_j\|^2 - 4d^2$.
As $i,j \longrightarrow \infty$, the right hand side inequality tends to zero. Thus $\{x_i\}$ is a Cauchy Sequence and has a limit point x. Since K is closed, $x \in K$. Since norm is continuous, $\|x\| = d$. For uniqueness, observe by parallelogram law that if $\|x\| = \|y\| = d$ and $x \neq y$, then $\|\frac{1}{2}(x+y)\| < d$

This completes the proof.

(3) <u>Statement</u> Let $\{f_1, f_2, \cdots\}$ be a linearly independent set of vectors in an inner product space. For each n, it is possible to define a vector g_n as a linear combination of $f_1, f_2, \cdots f_n$ in such a way that $\{g_1, g_2, \cdots\}$ is orthonormal.

<u>Proof</u>. By Induction, $g_1 = \frac{f_1}{\|f_1\|}$. Then g_1 is an orthonormal set. Of course $f_1 \neq 0$. Thus by induction $g_1, g_2, \cdots g_{n-1}$ has been defined. That is $\{g_1, g_2, \cdots g_{n-1}\}$ is orthonormal. for each $k<n$, g_k is a linear combination of $\{f_1, f_2, \cdots f_{n-1}\}$.
Define $u_n = f_n - \sum_{k<n} \langle f_n, g_k\rangle g_k$. Indeed for $i<n$, we have

$$\langle u, g_i\rangle = \langle f_n, g_i\rangle - \sum_{k<n} \langle f_n, g_k\rangle \langle g_k, g_i\rangle$$

$= \langle f_n, g_i \rangle - \sum_{k<n} \langle f_n, g_k \rangle \delta_{ki} = 0$

from the definition of u and from induction, u is linear combination of $f_1, f_2, \cdots f_n$. finally we note that $u \neq 0$; for if $u = 0$, then f_n must be a linear combination of $g_1, g_2, \cdots g_{n-1}$ and thus indirectly a linear combination of $f_1, f_2, \cdots f_{n-1}$ contradicting independence of f_i. Hence we may set

$g_n = \frac{u}{\|u\|}$ to make $\{g_1, g_2, \cdots g_n\}$ is an orthonormal set.

$$g_n = \frac{f_n - \sum_{i=1}^{n-1} \langle f_n, g_i \rangle g_i}{\| f_n - \sum_{i=1}^{n-1} \langle f_n, g_i \rangle g_i \|}$$

Hence proved

(4) The proof is by Induction

$$\begin{vmatrix} 1 & x_0 \\ 1 & x_1 \end{vmatrix} = (x_1 - x_0)$$

Now
$$\begin{vmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{vmatrix} = (x_2 - x_1)(x_2 - x_0)(x_1 - x_0)$$

$\vdots$

Proceeding in the similar way

$$\begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^2 \\ 1 & x_2 & x_2^2 & \cdots & x_2^2 \\ \vdots & & & & \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{vmatrix} = \prod_{0 \le i < j \le n} (x_i - x_j)$$

This Vandermonde's determinant of order n.

(b) We say that a set K is convex if $\sum_{i=1}^{m} \theta_i f_i \in K$ whenever $f_i \in K$, $\theta_i \geq 0$, $\sum_{i=1}^{m} \theta_i = 1$, $i = 1, 2, \ldots m$

Now consider

$$\{f_i : \|\sum_{i=1}^{m} \theta_i f_i - g\| = \{f_i : \|\sum_{i=1}^{m} \theta_i f_i - \sum_{i=1}^{m} \theta_i g\|\}$$

$$\Rightarrow \{f_i : \|\theta_1(f_1 - g) + \theta_2(f_2 - g) + \cdots + \theta_m(f_m - g)\}\|$$

$$\Rightarrow \{\cancel{f_i} \leq \theta_1\|f_1 - g\| + \theta_2\|f_2 - g\| + \cdots + \theta_m\|f_m - g\|\}$$

$$\Rightarrow \{\cancel{f_i} \leq r \sum_{i=1}^{m} \theta_i\}$$

$$\Rightarrow \{f_i : \|\sum_{i=1}^{m} \theta_i f_i - g\| \leq r\} \Rightarrow \sum_{i=1}^{m} \theta_i f_i \in K$$

Thus it is convex.

Proved

(5) (a) Let $x_0, x_1, \ldots x_n$ be the points and $y_0, y_1, \ldots y_n$ the prescribed values. We seek a polynomial p such that $P(x_i) = y_i$ $(i = 0, 1, \ldots n)$. Since the polynomial is of degree $\leq n$, it may be expressed as $P(x) = \sum_{j=0}^{n} c_j x^j$. Hence our requirement is that $\sum_{j=0}^{n} c_j x_i^j = y_i$ $(i = 0, \ldots n)$.

It can be written in the form of matrix

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \cdot \\ \cdot \\ \cdot \\ c_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix}$$

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In this equation, $c's$ are unknown where x matrix and right hand side are known. This equation has a unique solution because the coefficient matrix is non-singular. The determinant of this matrix, has the value

$$D = \prod_{0 \leq i < j \leq n} (x_i - x_j) \quad \text{——} \; (2)$$

The right hand side of (2) denotes the product of all factors $(x_i - x_j)$ for which the pair (i,j) satisfies $0 \leq j < i \leq n$. From the formula for D, it is clear that $D \neq 0$ if and only if the points x_i are distinct.

This completes the proof.

(b) For $f \in C[0,\infty)$, the Szâsz operator $\{S_n f\}$ are defined as

$$(S_n f)(x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{\lfloor k} f\left(\frac{k}{n}\right). \quad \text{——} \; (1)$$

We know that by expansion

$$e^{nx} = \sum_{k=0}^{\infty} \frac{(nx)^k}{\lfloor k}$$

$$\Rightarrow 1 = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{\lfloor k} \quad \text{——} \; (2)$$

Thus

$$|(S_n f)(x)| \le e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \left|f\left(\frac{k}{n}\right)\right|$$

$$\Rightarrow |(S_n f)(x)| \le \sum_{k=0}^{\infty} e^{-nx} \frac{(nx)^k}{k!} \|f\|$$

Using Σ, we get

$$\|S_n f\| \le \|f\|.$$

This completes.

8. For $f \in C[0,\infty)$, let the Szâsz Operator $\{L_n f\}$ be defined as

$$(L_n f)(x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{\underline{|k}} f\left(\frac{k}{n}\right)$$

Then we have to prove that

$$(L_n 1)(x) \xrightarrow[\text{uniformly}]{} 1 \quad \text{as } n \to \infty$$

$$(L_n t)(x) \xrightarrow{\text{uniformly}} x \quad \text{as } n \to \infty$$

$$(L_n t^2)(x) \xrightarrow{\text{uniformly}} x^2 \quad \text{as } n \to \infty$$

Then we state Korovkin Theorem on Monotone Operators $\Longrightarrow$ The proof of Weierstrass Approximation Theorem follows.

~~7~~ Let $T_{n,m}(x) = \sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^m \binom{n}{k} x^k (1-x)^{n-k}$ —— ①

diff ① w.r.t x, we get

$$T_{n,m}'(x) = \sum_{k=0}^{n} -m\left(\frac{k}{n}-x\right)^{m-1} \binom{n}{k} x^k (1-x)^{n-k} + \sum_{k=0}^{n} \left(\frac{k}{n}-x\right)^m k x^{k-1} (1-x)^{n-k} +$$

$$- \sum_{k=0}^{n} \binom{n}{k} \left(\frac{k}{n}-x\right)^m x^k (n-k)(1-x)^{n-k-1}$$

After simplification, we get

$$T_{n,m}'(x) = -m\, T_{n,m-1}(x) + n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{k}{n}-x\right)^{m+1} x^{k-1} (1-x)^{n-k-1}$$

Multiplying by $x(1-x)$, we get

$$x(1-x)\, T_{n,m}'(x) = -m x(1-x)\, T_{n,m-1}(x) + n\, T_{n,m+1}(x)$$

$$\Rightarrow n\, T_{n,m+1}(x) = x(1-x)\left[m\, T_{n,m-1}(x) + T_{n,m}'(x)\right]$$

with $T_{n,0}(x) = 1$

$$T_{n,1}(x) = 0 \Rightarrow T_{n,1}'(x) = 0$$

Putting $m=1$, we get

$$n\, T_{n,2}(x) = x(1-x)\left[1.\, T_{n,0}(x) + T_{n,1}'(x)\right]$$

$$\therefore T_{n,2}(x) = \frac{x(1-x)}{n} \Rightarrow T_{n,2}'(x) = \frac{1-2x}{n}$$

Putting $m=2$, we get

$$n\, T_{n,3}(x) = x(1-x)\left[2\, T_{n,1}(x) + T_{n,2}'(x)\right]$$

$$\Rightarrow T_{n,3}(x) = \frac{x(1-x)}{n}\left[2.0 + \frac{1-2x}{n}\right]$$

$$= \frac{x(1-x)(1-2x)}{n^2}$$

The value of $T_{n,4}(x)$ and $T_{n,4}'(x)$ may be done.
Proceeding in the same way, we get
$m=4$

$$n\, T_{n,5}(x) = x(1-x)\left[T_{n,\cancel{4}3}(x) + T_{n,4}'(x)\right]$$

$$\therefore T_{n,5}(x) = \sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^5 \binom{n}{k} x^k (1-x)^{n-k}$$

$$= \frac{x(1-x)}{n}\left[T_{n,3}(x) + T_{n,4}'(x)\right]$$

(8)

Now

$$g(x) - (B_n g)(x)$$

$$= \int_0^x f''(t)(x-t)\,dt - \sum_{k=0}^{n}\binom{n}{k} x^k (1-x)^{n-k} g\left(\frac{k}{n}\right)$$

$$= \int_0^x f''(t)(x-t)\,dt - \sum_{k=0}^{n}\binom{n}{k} x^k (1-x)^{n-k} \int_0^{\frac{k}{n}} f''(t)\left(\frac{k}{n}-t\right)dt$$

$$= \left[f'(t)(x-t)\right]_0^x + \int_0^x f'(t)\,dt - \sum_{k=0}^{n}\binom{n}{k} x^k (1-x)^{n-k}\left[f'(t)\left(\frac{k}{n}-x\right)\right]_0^{k}$$

$$+ \int_0^{k/n} f'(t)\,dt$$

$$= -f'(0)x + f(x) - f(0) - \sum_{k=0}^{n} p_{n,k}(x) f'\left(\frac{k}{n}\right)\left(\frac{k}{n} - \frac{k}{n}\right)$$

$$+ \sum_{k=0}^{n} \binom{n}{k} x^{k}(1-x)^{n-k} f'(0)\cdot\frac{k}{n} - \sum_{k=0}^{n} \binom{n}{k} x^{k}(1-x)^{n-k} f\left(\frac{k}{n}\right)$$

$$+ \sum_{k=0}^{n} p_{n,k}(x) f(0)$$

$$= F - B_n f.$$

Proved

12/8/14

I/c Dean
Faculty of Math &
Comp. Sci.